Behavior of Two-Stream Instability in Magnetic Field

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- The dispersion relation is

$$1 - \sum_\alpha \frac{\omega_{p\alpha}^2}{\left(\omega - \mathbf{k}\cdot\mathbf{v}_\alpha^{(0)}\right)^2} = 0,$$

where $\omega_{p\alpha} = \sqrt{\frac{n_\alpha q_\alpha^2}{\varepsilon_0 m_\alpha}}$ is the plasma frequency.

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Assumption for this model

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- We suppose that a solution of equations can be expanded about an equilibrium.

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- Especially, we are interested in imaginary part of the dispersion relation $\omega(\mathbf{k})$ which leads to an instability.

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Quantities in this model

- streaming velocity $\mathbf{v}_\alpha$,
- particle density n_α,
- particle mass m_α,
- particle charge q_α,
- temperature T_α,
- polytropic coefficient γ_α,
- electromagnetic field $\mathbf{E}, \mathbf{B}$,

where subscript $\alpha = 1, 2$ describe the first or the second stream.

Equations describing fluid model

- Continuity equation

$$\frac{\partial n_\alpha}{\partial t} + \nabla \cdot (n_\alpha \mathbf{v}_\alpha) = 0.$$

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- Equation of motion for the fluid

$$m_\alpha n_\alpha \frac{d\mathbf{v}_\alpha}{dt} = -\nabla p_\alpha + q_\alpha n_\alpha (\mathbf{E} + \mathbf{v}_\alpha \times \mathbf{B}).$$

Equations describing fluid model

- Equation of motion for the electromagnetic field

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{E}}{\partial t},$$

$$\frac{1}{\varepsilon_0 \mu_0} \nabla \times \mathbf{B} = \frac{1}{\varepsilon_0} \mathbf{j} + \frac{\partial \mathbf{E}}{\partial t}.$$

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- Equation for the pressure

$$p_\alpha = K_\alpha n_\alpha^{\gamma_\alpha},$$

where K_α is constant.

Linearization of the equations – expansion at a small parameter λ

- $n_\alpha = n_\alpha^{(0)} + \lambda n_\alpha^{(1)} + \mathcal{O}\left(\lambda^2\right)$,
- $\mathbf{v}_\alpha = \mathbf{v}_\alpha^{(0)} + \lambda \mathbf{v}_\alpha^{(1)} + \mathcal{O}\left(\lambda^2\right)$,
- $\mathbf{E} = \mathbf{E}^{(0)} + \lambda \mathbf{E}^{(1)} + \mathcal{O}\left(\lambda^2\right)$,
- $\mathbf{B} = \mathbf{B}^{(0)} + \lambda \mathbf{B}^{(1)} + \mathcal{O}\left(\lambda^2\right)$,
- $p_\alpha = p_\alpha^{(0)} + \lambda p_\alpha^{(1)} + \mathcal{O}\left(\lambda^2\right)$.

Linearized equations – Fourier picture

- $\Omega_\alpha n_\alpha^{(1)} - n_\alpha^{(0)}\mathbf{k}\cdot\mathbf{v}_\alpha^{(1)} = 0,$
- $-im_\alpha n_\alpha^{(0)}\Omega_\alpha \mathbf{v}_\alpha^{(1)} = -i\mathbf{k}p_\alpha^{(1)} + q_\alpha n_\alpha^{(0)}\mathbf{E}^{(1)} + q_\alpha n_\alpha^{(1)}\mathbf{E}^{(0)} + q_\alpha n_\alpha^{(0)}\mathbf{v}_\alpha^{(0)} \times \mathbf{B}^{(1)} + q_\alpha n_\alpha^{(0)}\mathbf{v}_\alpha^{(1)} \times \mathbf{B}^{(0)} + q_\alpha n_\alpha^{(1)}\mathbf{v}_\alpha^{(0)} \times \mathbf{B}^{(0)},$
- $i\mathbf{k}\times\mathbf{E}^{(1)} = i\omega\mathbf{B}^{(1)},$
- $-i\omega\mathbf{E}^{(1)} = \frac{i}{\varepsilon_0\mu_0}\mathbf{k}\times\mathbf{B}^{(1)} - \sum_{\alpha=1}^{2}\frac{q_\alpha}{\varepsilon_0}(n_\alpha^{(1)}\mathbf{v}_\alpha^{(0)} + n_\alpha^{(0)}\mathbf{v}_\alpha^{(1)}),$
- $p_\alpha^{(1)} = m_\alpha c_\alpha^2 n_\alpha^{(1)}.$

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- Next we will consider low frequency limit $\frac{\omega}{c} \ll k$.

Equations for the perturbation velocities

- $\Omega_1^2 \mathbf{v}_1^{(1)} = i\frac{q_1}{m_1}(\mathbf{k}\cdot\mathbf{v}_1^{(1)})[\mathbf{E}^{(0)} + \mathbf{v}_1^{(0)} \times \mathbf{B}^{(0)}] + (\mathbf{k}\cdot\mathbf{v}_1^{(1)})[c_1^2 + \frac{\omega_{p1}^2}{k^2}]\mathbf{k} + \frac{q_1\Omega_1 m_2 \omega_{p2}^2}{q_2\Omega_2 m_1 k^2}(\mathbf{k}\cdot\mathbf{v}_2^{(1)})\mathbf{k} + i\frac{q_1\Omega_1}{m_1}\mathbf{v}_1^{(1)} \times \mathbf{B}^{(0)}$,

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- $\Omega_2^2 \mathbf{v}_2^{(1)} = i\frac{q_2}{m_2}(\mathbf{k}\cdot\mathbf{v}_2^{(1)})[\mathbf{E}^{(0)} + \mathbf{v}_2^{(0)} \times \mathbf{B}^{(0)}] + (\mathbf{k}\cdot\mathbf{v}_2^{(1)})[c_2^2 + \frac{\omega_{p2}^2}{k^2}]\mathbf{k} + \frac{q_2\Omega_2 m_1 \omega_{p1}^2}{q_1\Omega_1 m_2 k^2}(\mathbf{k}\cdot\mathbf{v}_1^{(1)})\mathbf{k} + i\frac{q_2\Omega_2}{m_2}\mathbf{v}_2^{(1)} \times \mathbf{B}^{(0)}$,

where $\omega_{p\alpha}^2 = \frac{q_\alpha^2 n_\alpha^{(0)}}{\varepsilon_0 m_\alpha}$, $\Omega_\alpha = \omega - \mathbf{k}\cdot\mathbf{v}_\alpha^{(0)}$, $k^2 = \mathbf{k}\cdot\mathbf{k}$, $\alpha = 1, 2$.

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- We try to find a solution of the homogeneous system of linear equations.
- The homogeneous system of linear equations has a solution if and only if the determinant of the system is equal to zero.
- Then the dispersion relation $\omega(\mathbf{k})$ is obtained from a condition for a nontrivial solution.
- This condition leads to the determinant of the matrix 6x6.

Analytical solution?

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- We get a polynomial in ω and we want to find its roots. The roots are just the dispersion relations.
- Unfortunately, it is difficult for analytical solving because the degree of the polynomial is eight.

Explicit analytical solution

$$\Omega_1^4 + \Omega_1^2[\frac{i}{m_1}\mathbf{F}_1^{(0)} \cdot \mathbf{k} + (c_1^2 + \frac{\omega_{p1}^2}{k^2})k^2 + \frac{q_1^2}{m_1^2}\mathbf{B}^{(0)2}] + \Omega_1 \frac{q_1}{m_1^2}(\mathbf{F}_1^{(0)} \times \mathbf{k}) \cdot \mathbf{B}^{(0)} - \frac{q_1^2}{m_1^2}(\mathbf{k} \cdot \mathbf{B}^{(0)})[\frac{iq_1}{m_1}\mathbf{E}^{(0)} \cdot \mathbf{B}^{(0)} + (c_1^2 + \frac{\omega_{p1}^2}{k^2})\mathbf{k} \cdot \mathbf{B}^{(0)}] - \frac{\omega_{p1}^2\omega_{p2}^2}{k^4}[\Omega_1^2 k^2 - \frac{q_1^2}{m_1^2}(\mathbf{B}^{(0)} \cdot \mathbf{k})][\Omega_2^2 k^2 - \frac{q_2^2}{m_2^2}(\mathbf{B}^{(0)} \cdot \mathbf{k})][\Omega_2^4 + \Omega_2^2[\frac{i}{m_2}\mathbf{F}_2^{(0)} \cdot \mathbf{k} + (c_2^2 + \frac{\omega_{p2}^2}{k^2})k^2 + \frac{q_2^2}{m_2^2}\mathbf{B}^{(0)2}] + \Omega_2 \frac{q_2}{m_2^2}(\mathbf{F}_2^{(0)} \times \mathbf{k}) \cdot \mathbf{B}^{(0)} - \frac{q_2^2}{m_2^2}(\mathbf{k} \cdot \mathbf{B}^{(0)})(\frac{iq_2}{m_2}\mathbf{E}^{(0)} \cdot \mathbf{B}^{(0)} + (c_2^2 + \frac{\omega_{p2}^2}{k^2})\mathbf{k} \cdot \mathbf{B}^{(0)})]^{-1} = 0$$

Open problems

- Numerical solutions of the dispersion relation. (It is important to know the magnetic field in plasma for numerical solving.)

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- Generalization problem for an arbitrary frequency (we have only low frequency model).
- Solution with a boundary condition.